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ON THE GENERALIZATION OF A THEOREM IN SOLID GEOMETRY.

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It is shown in solid geometry that the volume of a truncated right triangular prism is equal to the product of the area of its base by one-third the sum of the lateral edges. In symbols,

$$V_3 = B_3 \left(\frac{e_1 + e_2 + e_3}{3} \right), \quad [1]$$

where B_3 denotes the base; e_1, e_2, e_3 the lateral edges; and V_3 the volume of the truncated prism.

A truncated right quadrangular prism may be divided into two truncated right triangular prisms by passing a plane through a pair of opposite lateral edges, say through e_1 and e_3 , where e_1, e_2, e_3, e_4 denote the lateral edges taken in succession. If furthermore B_4 denotes the quadrangular base, B'_3 and B''_3 the respective bases of the component triangular prisms, and V_4 the volume of the truncated quadrangular prism, the application of formula [1] gives

$$V_4 = B'_3 \left(\frac{e_1 + e_2 + e_3}{3} \right) + B''_3 \left(\frac{e_3 + e_4 + e_1}{3} \right). \quad [2]$$

If the base of the truncated prism is a parallelogram, B'_3 and B''_3 are each equal to one-half of B_4 , [2] therefore becomes

$$V_4 = \frac{B_4}{2} \left(\frac{e_1 + e_2 + e_3 + e_4 + e_1 + e_3}{3} \right). \quad [3]$$

But now the upper base is a parallelogram also, and from the fact that the diagonals of this upper base bisect each other it readily follows that

$$e_1 + e_3 = e_2 + e_4,$$

from which

$$e_1 + e_3 = \frac{e_1 + e_2 + e_3 + e_4}{2}$$

By substituting this value of $e_1 + e_3$, formula [3] reduces to

$$V_4 = B_4 \left(\frac{e_1 + e_2 + e_3 + e_4}{4} \right), \quad [4]$$

that is,

The volume of a truncated right quadrangular prism, whose base is a parallelogram, is equal to the product of the area of its base by one-fourth the sum of the lateral edges.

The question naturally suggests itself whether this theorem may be generalized so as to apply to a truncated prism of any number of faces. The answer is that it can be, provided the prism is regular. In that case, the corresponding formula for the volume is

$$V_n = B_n \left(\frac{e_1 + e_2 + e_3 + \dots + e_n}{n} \right). \quad [5]$$

The writer is not aware that a proof of this theorem has ever been published, nor does the theorem appear to have received even mention in text-books on solid geometry. The following proof is of interest not only as a simple proof of the theorem in question, but also as furnishing an elegant application of a number of trigonometric principles.

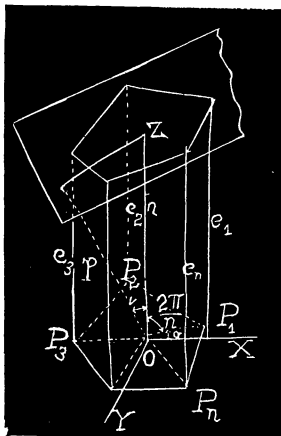
Let B_n denote the base of a truncated regular prism of n sides, $P_1, P_2, P_3, \dots, P_n$ the vertices of the base, and $e_1, e_2, e_3, \dots, e_n$ the corresponding edges.

Through each of the edges and O the center of the base pass planes dividing the solid into n truncated triangular prisms. The bases of these component prisms are equal and each equal to $\frac{B_n}{n}$. Let h represent the edges common to all the

triangular prisms. Formula [1] then gives us for

the volume of the sum of the truncated triangular prisms,

$$V_n = \frac{B_n}{n} \left(\frac{e_1 + e_2 + h}{3} + \frac{e_2 + e_3 + h}{3} + \dots + \frac{e_n + e_1 + h}{3} \right)$$



$$= \frac{B_n}{n} \left[\frac{2(e_1 + e_2 + e_3 + \dots + e_n) + nh}{3} \right] \quad [6]$$

Now take the plane of the base of the prism for the xy -plane of coordinates, and a perpendicular at O for the z -axis. Let ρ, a denote the polar coordinates of the vertex P_1 with reference to O as pole and OX as initial line. The central angles subtended by the sides of the polygon are all equal and each equal to $\frac{2\pi}{n}$, hence the rectangular coordinates of the successive vertices are

$$\text{for } P_1, x_1 = \rho \cos a, \quad y_1 = \rho \sin a,$$

$$\text{for } P_2, x_2 = \rho \cos\left(a + \frac{2\pi}{n}\right), \quad y_2 = \rho \sin\left(a + \frac{2\pi}{n}\right),$$

$$\text{for } P_3, x_3 = \rho \cos\left(a + \frac{4\pi}{n}\right), \quad y_3 = \rho \sin\left(a + \frac{4\pi}{n}\right),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\text{for } P_n, x_n = \rho \cos\left(a + \frac{2(n-1)\pi}{n}\right), \quad y_n = \rho \sin\left(a + \frac{2(n-1)\pi}{n}\right).$$

Let

$$x \cos \lambda + y \cos \mu + z \cos \nu - p = 0$$

be the equation of the plane of the upper base of the truncated prism, then

$$z = \frac{p - x \cos \lambda - y \cos \mu}{\cos \nu}$$

The length of any edge e_{k+1} is found by substituting in z for x and y the coordinates x_{k+1}, y_{k+1} of the corresponding vertex of the base; hence,

$$e_{k+1} = \frac{p - \rho \cos \lambda \cos\left(a + \frac{2k\pi}{n}\right) - \rho \cos \mu \sin\left(a + \frac{2k\pi}{n}\right)}{\cos \nu}$$

If we develop $\cos\left(a + \frac{2k\pi}{n}\right)$ and $\sin\left(a + \frac{2k\pi}{n}\right)$ and combine the resulting terms in $\cos \frac{2k\pi}{n}$ and $\sin \frac{2k\pi}{n}$, respectively, we obtain

$$e_{k+1} = \frac{p}{\cos \nu} - \frac{\rho (\cos a \cos \lambda - \sin a \cos \mu)}{\cos \nu} \cos \frac{2k\pi}{n} \\ - \frac{\rho (\sin a \cos \lambda + \cos a \cos \mu)}{\cos \nu} \sin \frac{2k\pi}{n}.$$

But p is the perpendicular distance from O to the plane of the upper base and ν is the angle between this perpendicular and OZ or h , hence $\frac{p}{\cos \nu} = h$. Again, the coefficients of $\cos \frac{2k\pi}{n}$ and $\sin \frac{2k\pi}{n}$ do not involve k , they are the same for every edge, hence denoting them by A and B , respectively, we have

$$e_{k+1} = h + A \cos \frac{2k\pi}{n} + B \sin \frac{2k\pi}{n}$$

and we obtain for the sum of all the edges

$$\begin{aligned} e_1 + e_2 + e_3 + \dots + e_n &= nh \\ &+ A \left(\cos 0 + \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} \right), \\ &+ B \left(\sin 0 + \sin \frac{2\pi}{n} + \sin \frac{4\pi}{n} + \dots + \sin \frac{2(n-1)\pi}{n} \right). \end{aligned} \quad [7]$$

The series in the parentheses of [7] may be summed, as follows:

$$\begin{aligned} 2 \cos 0 \cdot \sin \frac{\pi}{n} &= \sin \frac{\pi}{n} + \sin \frac{\pi}{n}, \\ 2 \cos \frac{2\pi}{n} \cdot \sin \frac{\pi}{n} &= \sin \frac{3\pi}{n} - \sin \frac{\pi}{n}, \\ 2 \cos \frac{4\pi}{n} \cdot \sin \frac{\pi}{n} &= \sin \frac{5\pi}{n} - \sin \frac{3\pi}{n}, \\ &\vdots \\ 2 \cos \frac{2(n-1)\pi}{n} \cdot \sin \frac{\pi}{n} &= \sin \frac{(2n-1)\pi}{n} - \sin \frac{(2n-3)\pi}{n}. \end{aligned}$$

Adding,

$$2 \sin \frac{\pi}{n} \left(\cos 0 + \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} \right) = \sin \frac{\pi}{n} + \sin \frac{(2n-1)\pi}{n}$$

But

$$\sin \frac{\pi}{n} + \sin \frac{(2n-1)\pi}{n} = 2 \sin \frac{2n\pi}{2n} \cos \frac{(2n-2)\pi}{2n} = 0,$$

$$\text{since } \sin \frac{2n\pi}{2n} = \sin \pi = 0,$$

hence

$$\cos 0 + \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} = 0. \quad [8]$$

Again,

$$2\sin 0 \cdot \sin \frac{\pi}{n} = 0,$$

$$2\sin \frac{2\pi}{n} \cdot \sin \frac{\pi}{n} = \cos \frac{\pi}{n} - \cos \frac{3\pi}{n},$$

$$2\sin \frac{4\pi}{n} \cdot \sin \frac{\pi}{n} = \cos \frac{3\pi}{n} - \cos \frac{5\pi}{n},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$2\sin \frac{2(n-1)\pi}{n} \cdot \sin \frac{\pi}{n} = \cos \frac{(2n-3)\pi}{n} - \cos \frac{(2n-1)\pi}{n}.$$

Adding,

$$\begin{aligned} 2\sin \frac{\pi}{n} \left(\sin 0 + \sin \frac{2\pi}{n} + \sin \frac{4\pi}{n} + \dots + \sin \frac{2(n-1)\pi}{n} \right) \\ = \cos \frac{\pi}{n} - \cos \frac{(2n-1)\pi}{n} = 2\sin \pi \sin \frac{(n-1)\pi}{n} = 0, \end{aligned}$$

hence, also,

$$\sin 0 + \sin \frac{2\pi}{n} + \sin \frac{4\pi}{n} + \dots + \sin \frac{2(n-1)\pi}{n} = 0. \quad [9]$$

With the values [8] and [9] equation [7] becomes

$$e_1 + e_2 + e_3 + \dots + e_n = nh,$$

and this value of nh put in [6] gives

$$V_n = B_n \left(\frac{e_1 + e_2 + e_3 + \dots + e_n}{n} \right),$$

that is,

The volume of any truncated regular prism is equal to the product of the area of its base by the arithmetic mean of the edges.

As a limiting case of this theorem we obtain the well known rule for finding the volume of a truncated right circular cylinder.

The above proof could have been somewhat shortened, by choosing the x -axis so as to pass through one of the vertices of the base of the truncated prism. In that case $a=0$ from the outset.